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AN APPROACH TO THE POLYGONAL KNOT PROBLEM USING PROJECTIONS AND ISOTOPIES

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Abstract. The author extends earlier work of Tait, Gauss, Nagy, and Penney in defining and developing properties of (1) the boundary collection of a knot function, and (2) simple sequences of knot functions or boundary collections. The main results are (1) if two knot functions have isomorphic boundary collections then the knots they determine are equivalent, and (2) if two knot functions determine equivalent knots, then the given functions (their boundary collections) are the ends of a simple sequence of knot functions (boundary collections). Matrices are also defined for knot functions.

1. **Introduction.** In [15] Tait considered the idea of associating with an oriented polygonal knot M in regular position a "word" $W(M)$. For a trefoil knot whose projection double points are in order a, b, c such a word could be $ab^{-1}ca^{-1}bc^{-1}$. Tait observed various properties of such words, and used them as an aid in developing quite an extensive set of knot tables. For example, he utilized the fact that between a letter and its inverse there must be an even number, perhaps zero, of other letters. He also recognized that the complementary domains of a knot projection can be colored in a "checker board" pattern (see [16]).

Earlier Gauss in [3] had studied similar finite sequences in connection with knot projections and had actually conjectured the even number property. In [8] Nagy proved Gauss' conjecture, and observed that a knot projection could be colored alternately with two colors a, b such that as a point moved from one section through a crossing point into another section, the colors of the sections changed. Nagy also defined simple closed curves of the same color to be *cycles*.

In [10] Penney studied "words" for knots and defined certain "admissible" operations on words. He showed that if an admissible operation is performed on a word W of a knot K to form a new word W' then there is an equivalent knot K' whose word is W'. Penney also proved an isomorphism theorem for a pair of knots, where one of them has a prime word, and the other's word can be changed by a finite sequence of admissible operations to an isomorphic word. A word W is prime if W cannot be written as ABC where B and $A \cup C$ are nonvoid, and where if $x^e \in B$ then $x^{-e} \in B$.

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In [17] Treybig shows that if knots have isomorphic prime words then there is a certain unique correspondence between the complementary domains of their projections. In [16] Treybig characterizes those words in $2n$ letters that are words for knot projections. Marx uses a condition of Mac Lane to give a simpler characterization in [5].

In the present paper the author combines the ideas of Penney's paper above and his own characterization paper, and associates with an oriented knot in regular position not only a word but an object called a "boundary collection". A notion of isomorphism and also "admissible" operations are defined for these. The first theorem says that if two knots K, L have isomorphic boundary collections, then they are isomorphic in the sense that there is a piecewise linear homeomorphism of E^3 onto E^3 taking K onto L. Theorem 8 yields that if two knots K, L are isomorphic under an orientation-preserving homeomorphism of the space, then one may be obtained from the other by a certain finite sequence of "simple" moves and a corresponding sequence is obtained for the boundary collections. The importance of the boundary collection is that nearly all essential information about a given knot is "stored" in this object. Such collections can be given matrix representations, so it is evident that these would be highly useful in a study of knots by computer for example.

The main obstacle in the path of making this approach to knots complete is in not being able to show something like the following conjecture:

"If a piece of string in the shape of a simple closed curve can be moved in space so as to form at different times two knots in regular position whose projections have no more than n crossing points, then the string can be moved from one configuration to the other in small steps so that no projection has more than $2n$ crossing points."

In a later paper the author has some partial results on this problem, but no solution. The problem seems very difficult.

A student of the author, John Martin, has been able to show that if the above conjecture is true, then a modification of the ideas above yields a complete solution to various problems about invertible knots, amphicherial knots, etc. He also obtains certain results in connection with the word problem for Wirtinger presentations.

2. **Definitions and notation.** A *polygonal knot* (=*knot*) is a polyhedral simple closed curve in E^3. A knot K is in *regular position relative to a* plane W provided K is a union of straight line intervals $A_1A_2, A_2A_3, \ldots, A_nA_1$, where the projection $\pi_w(K)$ into W is at most two-to-one, and $x \in K$ and $\pi_w(x) = \pi_w(A_p)$ implies $x = A_p$. The points where the projection is two-to-one are *crossing points* or *double points* of the projection. K in regular position means relative to the xy plane. K written as $A_1A_2 \cdots A_n$ means each A_pA_{p+1} is a straight line interval (if $p=n$, let $p+1 \equiv 1$), and card $(A_pA_{p+1} \cap A_qA_{q+1}) > 1$ implies $p=q$.

If f: $[0, 1) \to K$ is continuous, one-to-one, and onto, with K a knot, then f is a *knot function*. Given f: $[0, 1) \to K$ a knot function, K in regular position and $0 \leqq a_1 < a_2 < \cdots < a_{2n} < 1$ the elements x of $[0, 1)$ so that $\pi f(x)$ is a double point of $\pi(K)$, then the a_i are called *double points of f*, and if $\pi f(a_i) = \pi f(a_j)$ $(i \neq j)$ let $a_i = m(a_j)$ (the mate). Under the above conditions *the word of f* is

$$W(f) = (\pi f(a_1))^{e_1}(\pi f(a_2))^{e_2} \cdots (\pi f(a_{2n}))^{e_{2n}}$$

where if $a_j = m(a_i)$ then $e_i = 1$ (or is omitted) and $e_j = -1$ if $f(a_i)$ has a larger z coordinate than $f(a_j)$. To define the *word of the knot function f*: $[0, 1) \to K$ *relative to plane P and a given side S* of P assume K in regular position relative to P and define in the obvious way making use of a transformation T having the property that $T(P)$ is the xy plane and the positive z axis is a subset of $T(S)$, where T is a rotation of E^3 followed by a translation.

Given f: $[0, 1) \to K$ a knot function, with K in regular position, the *boundary collection C(f) of f* is $\{D$: there is a complementary domain U of πK so that $d \in D$ provided $d = (\pi f(a_j))^{e_i}(\pi f(a_{i+1}))^{e_{i+1}}$, where $\pi f([a_i, a_{i+1}]) \subset \mathrm{Bd}(U)\}$. Let $[a_{2n}, a_1]$ denote $[a_{2n}, 1) \cup [0, a_1]$. We extend the above definition to define *boundary collection C(f) of f relative to plane P and a given side S* of P.

If (1) f: $[0, 1) \to K$ and g: $[0, 1) \to L$ are knot functions with K, L in regular position and (2) $m, n \in \{0, 1, 2, 3, \ldots\}$ then g *has been obtained from f by an m by n transformation T* (write $f \xrightarrow{T} g$) provided there exists $S \subset [0, 1)$ and a polyhedral disk D with triangulation T_D so that (0) π is 1-1 on D, (1) the simple closed curve $\mathrm{Bd}(D)$ is the union of arcs AXB, AYB, (2) $f(S) = AXB$ and $g(S) = AYB$, (3) πAXB (alternately, πAYB) contains exactly m (alternately, n) double points of πK (alternately, πL), (4) neither A nor B is a double point of πK nor of πL, (5) $K \cap D = AXB$ and $L \cap D = AYB$, (6) $f(x) = g(x)$ if $x \in [0, 1) - S$, and (7) if the interval G is a subset of $K - D$ and V is a 1-simplex in T then card $(\pi G \cap \pi V)$ is no more than 1.

We define some special types of m by n transformations.

(a) *Type* 0. This is where (1) $m = n$, (2) if C is a component of $\pi K \cap \mathrm{Int}(\pi(D))$, then $\bar{C} \cap (\pi \,\mathrm{Bd}(D)) = \{\pi x, \pi y\}$ where $x \in \mathrm{seg}\, AXB$ and $y \in \mathrm{seg}\, AYB$ or $\bar{C} \cap (\pi \,\mathrm{Bd}(D)) = \{\pi A, \pi B\}$, and (3) there are no double points of πK in $\mathrm{Int}\, \pi(D)$.

(b) *Types* I *and* I′. When (1) there is exactly one component C of $\pi K \cap \mathrm{Int}\, \pi(D)$, where C contains no double point of πK, and (2) $\bar{C} \cap (\pi \,\mathrm{Bd}(D)) = \{\pi x, \pi y\}$ where $x = A$ or B and $y \in \mathrm{seg}\, AYB$, we say g (alternately f) *has been obtained from f* (alternately g) *by a Type* I (alternately Type I′) *transformation* $(m = 0, n = 1)$.

(c) *Types* II *and* II′. When (1) there is exactly one component C of $\pi K \cap \mathrm{Int}\, \pi(D)$, where C contains no double points of πK, and (2) both points of $\bar{C} - C$ lie in $\pi \,\mathrm{seg}\, AYB$, then we say g (alternately, f) *has been obtained from f* (alternately, g) *by a Type* II (II′) *transformation.*

(d) *Type* III. When (1) there is exactly one component C of $\pi K \cap \mathrm{Int}\, \pi(D)$ and there is exactly one crossing point of πK in C, and (2) $\bar{C} - C$ has exactly two points in π(seg AXB) and exactly two in π(seg AYB), then we say g has been obtained from f by a Type III transformation. Here $m = n = 2$.

Now suppose f: [0, 1) $\to K$ and g: [0, 1) $\to L$ are knot functions with words

$$W(f) = (\pi(f(a_1)))^{e_1}\cdots(\pi f(a_{2n}))^{e_{2n}}$$

and

$$W(g) = (\pi(g(b_1)))^{d_1}\cdots(\pi g(b_{2m}))^{d_{2m}}$$

respectively. We say $W(f)$ is isomorphic to $W(g)$ ($W(f)\cong W(g)$) if and only if (1) $m=n$, (2) there is a cyclic permutation h of $1,\ldots,2n$ so that $\pi f(a_i)=\pi f(a_j)$ if and only if $\pi g(b_{h(i)})=\pi g(b_{h(j)})$ and $d_{h(i)}=e_i$ and $d_{h(j)}=e_j$. We say $C(f)$ *is isomorphic to* $C(g)$ ($C(f)\cong C(g)$) if and only if (1) $W(f)\cong W(g)$ (with h as above) and (2) there is a 1-1 correspondence $K\to K'$ between the elements of $C(f)$ and those of $C(g)$ such that if $K\to K'$ then $(\pi f(a_p))^{e_p}(\pi f(a_{p+1}))^{e_{p+1}}\in K$ if and only if

$$\pi g(b_{h(p)})^{d_{h(p)}}(\pi g(b_{h(p+1)}))^{d_{h(p+1)}}\in K'.$$

The m by n transformation of Types 0, I, I′, II, II′, III are called *simple transformations.* Also if $f_0, f_1, \ldots, f_n$ is a sequence of knot functions and there exist simple transformations $T_1,\ldots,T_n$ so that $T_p: f_{p-1}\to f_p$ then we call

$$f_0 \xrightarrow{T_1} f_2 \xrightarrow{T_2} \cdots \xrightarrow{T_n} f_n$$

a *simple sequence of knot functions.* Furthermore, if $f_0, f_1, \ldots, f_n$ is a sequence of knot functions, we shall say $C(f_0)\xrightarrow{S_1} C(f_1)\to\cdots\xrightarrow{S_n} C(f_n)$ is a simple sequence of boundary collections provided it is true that, if $0\leqq p\leqq n-1$, then either (a) $C(f_p)\cong C(f_{p+1})$, or (b) there exists knot functions g, h so that $C(g)\cong C(f_p)$, $C(h)\cong C(f_{p+1})$, and h is obtained from g by a simple transformation T. In case (b) we also say that S_p is of the same type as T.

3. **A basic theorem.** In this section, we give a proof of the theorem which shows that the boundary collection of a knot function retains a great deal of information about the function.

THEOREM 1. *Suppose that each of* f: [0, 1) $\to K$ *and* g: [0, 1) $\to L$ *is a knot function. Then, if* $C(f)\cong C(g)$ *there is a piecewise linear* (*p.l.*) *homeomorphism* h *from* E^3 *onto* E^3 *such that* (1) $h(K)=L$, (2) *if* x *and* y *are distinct elements of* K *so that* $\pi(x)=\pi(y)$, *then interval* xy *has the property that* $h(xy)$ *is an interval from* $h(x)$ *to* $h(y)$, *and* (3) $h(f)$ *induces the same orientation on* L *as* g. *Furthermore, if the element of* $C(f)$ *corresponding to the unbounded complementary domain of* πK *also corresponds to the analogous element of* $C(g)$ *under the isomorphism* $C(f)\cong C(g)$, *then* h *may be chosen so that* $h(\pi x)=\pi h(x)$.

Proof. Following Penney in Lemma 4 of [10], let the double points of f be $a_1,\ldots,a_{2n}$ and those of g be $b_1,\ldots,b_{2n}$. Since $C(f)\cong C(g)$ implies that $W(f)\cong W(g)$, let us suppose that $a_1\sim b_1$ under this correspondence.

Let U denote the complementary domain of πK which corresponds to the unbounded complementary domain of πL under the correspondence $C(f) \cong C(g)$; and suppose the worst case, that U is bounded. There is a polygonal arc pq with p in the unbounded complementary domain of πK and q in U such that (1) pq contains no double points of πK, (2) if $S = [a_i, a_{i+1}]$ then $\pi f(S)$ contains at most one point of pq and (3) if V is a complementary domain of πK, then $V \cap pq$ is connected. Let $r_0, r_1, \ldots, r_{m-1}$ denote the points of $\pi K \cap pq$ in their natural order from p to q, and let $(c_0, d_0), \ldots, (c_{m-1}, d_{m-1})$ denote open subintervals of $[0, 1)$ so that $r_i \in \pi f((c_i, d_i))$ and

$$\mathrm{Cl}\left(\bigcup_{i=0}^{m-1} (c_i, d_i)\right) \subset [0, 1) - \{a_1, \ldots, a_{2n}\}.$$

Using Corollary 1 of Sanderson [12], we may construct knot functions $k_0 = f$, k_1, $\ldots, k_m$ and homeomorphisms $h_1, \ldots, h_m$ such that for each integer p, $0 \leqq p < m-1$,

(1) $k_{p+1}(x) = k_p(x)$ if $x \in [0, 1) - (c_p, d_p)$,

(2) $\pi(k_{p+1}([c_p, d_p]) \cup k_p([c_p, d_p]))$ is a polygonal simple closed curve containing $\pi k_p([0, 1) - (c_p, d_p))$ in its interior, and

(3) h_p is a p.l. homeomorphism from E^3 onto E^3 that takes $k_p[0, 1)$ onto $k_{p+1}[0, 1)$ in such a way that it is the identity on any straight line interval joining points of the form $k_p(a_i)$ and $k_p(a_j)$, where $m(a_i) = a_j$ $(i \neq j)$, and

(4) $W(k_0) = W(k_1) = \cdots = W(k_m)$ and $C(k_0) = C(k_1) = \cdots = C(k_m)$.

We now define a p.l. homeomorphism w of the xy plane onto itself. First consider triangulations R_1 and R_2 of $\pi k_m([0, 1))$ and πL, respectively, such that, if $1 \leqq i \leqq 2n$, then $\pi k_m(a_i)$ is a 0-simplex of R_1 and $\pi g(a_i)$ is a 0-simplex of R_2. Let t denote a p.l. homeomorphism from $\pi k_m([0, 1))$ onto πL so that, for each $i = 1, \ldots, 2n$, $t\pi k_m(a_i) = \pi g(a_i)$ and $t\pi k_m([a_i, a_{i+1}]) = \pi g([a_i, a_{i+1}])$.

Now suppose V is a complementary domain of $\pi k_m([0, 1))$ and V' is its corresponding domain under the correspondence $C(k_m) \cong C(g)$. We give the argument here in the case V is bounded, but the unbounded case is also easily handled. Let $d_1, \ldots, d_j$ denote the double points of $\pi k_m([0, 1))$ on Bd V, and let $d'_1, \ldots, d'_j$ denote those of πL on Bd V', where $\pi k_m(a_t) = d_s$ implies that $\pi g(a_t) = d'_s$. Now suppose $P \in V$ and consider polygonal arcs $PD_1, PD_2, \ldots, PD_N$ such that (a) $\{D_1, \ldots, D_N\} = \{d_1, \ldots, d_j\}$, (b) $PD_p - D_p \subset V$, $p = 1, \ldots, N$, (c) if $\pi k_m([a_i, a_{i+1}]) \subset \mathrm{Bd}\, V$ and $m(a_i) = a_{i+1}$, then there exist integers $p, p+1$ so that $\pi k_m((a_i, a_{i+1})) \subset \mathrm{Int}\,(PD_p \cup PD_{p+1})$, (d) if $\pi k_m([a_i, a_{i+1}]) \subset \mathrm{Bd}\, V$ and $m(a_i) \neq a_{i+1}$, then there exist integers $p, p+1$ so that $\{D_p, D_{p+1}\} = \{\pi k_m(a_i), \pi k_m(a_{i+1})\}$, (e) the arcs PD_p and PD_q $(p \neq q)$ intersect in no points besides endpoints, and (f) no $d_i = D_p$ for more that two indices p, and if d_i does not separate Bd V then $d_i = D_p$ for a unique p. We also pick P' and V' and a corresponding set of polygonal arcs $P'D'_1, \ldots, P'D'_n$ such that

(1) (a)–(f) hold for the primed quantities, and

(2) $D_p = d_i$ implies $D'_p = d'_i$.

Now let $t_V: (\text{Bd } V \cup \sum PD_p) \to (\text{Bd } V' \cup \sum P'D'_p)$ be an onto p.l. homeomorphism such that (a) $t_V(P)=P'$, $t_V(D_p)=D'_p$ $(p=1,\dots,N)$, and $t_V(PD_p)=P'D'_p$ $(p=1,\dots,N)$, and (b) t_V agrees with t on Bd V.

For each component C of $V-\sum PD_p$ bounded by two consecutive arcs PD_p, PD_{p+1} and an arc (or simple closed curve) D_pD_{p+1}, we extend our transformation t_V to a p.l. homeomorphism t_C carrying $\bar{C}$ onto the corresponding set in $\bar{V}'$. Each set $\bar{C}$ is either a polyhedral 2-cell or one with two boundary points identified, so the mapping is not hard to extend. The mappings t_C combine to define a p.l. homeomorphism $l_V: \bar{V} \to \bar{V}'$ so that $l_V(x)=t(x)$ if $x \in \text{Bd } V$.

We now define w from the xy plane onto the xy plane so that if x is an element of $\bar{V}$, where V is a complementary domain of $\pi k_m([0, 1))$, then $w(x)=l_V(x)$. Let T_1 denote a triangulation of the xy plane so that (1) w is linear on each simplex s of T_1, (2) each $\pi k_m(a_i)$ is a 0-simplex of T_1, (3) the 1-skeleton of T_1 contains $\pi k_m([0, 1))$, and (4) if $x \in k_m([0, 1))$ and is not a subset of an open interval lying in that set, then πx is a 0-simplex of T_1; and if $y \in g([0, 1))$ and is not a subset of an open interval lying in that set, then $w^{-1}(\pi(y))$ is a 0-simplex of T_1.

We now define a homeomorphism k from $k_m([0, 1))$ onto $g([0, 1))$. For each point $k_m(a_i)$ let k of $(k_m(a_i))=g_m(a_i)$. Thus on each line parallel to the z axis which contains two points of $k_m[0, 1)$, we have defined k for those two points, so we extend linearly. For every other such line which passes through a 0-simplex of T_1 and a point of $k_m[0, 1)$, say $k_m(x)$, let $k(k_m(x))=\pi^{-1}(w(x)) \cap L$ and let k on that line simply denote a translation. For every other such line through a zero simplex s of T_1 let $T_1(l)$ be a translation of l so that it contains $w(s)$. k is now defined on each line parallel to the z axis containing a 0-simplex of T_1. We extend the function now on each triangular based cylinder to find k. h is defined to be $kh_mh_{m-1}\cdots h_1$.

4. **Approximations.** In this section, we prove a theorem on approximations which is needed in the later sections.

Following Sanderson [12] we assume that all simplexes are closed and that if $V_0,\dots, V_n$ is a set of points, then $\langle V_0,\dots, V_n\rangle$ will denote the closed convex hull of $\{V_0,\dots, V_n\}$ (this is the n simplex whose vertices are $V_0,\dots, V_n$ provided $V_0,\dots, V_n$ is an affinely independent set).

THEOREM 2. *Suppose T is a finite m-complex in Euclidean n-space E^n. Then, there is a positive number ε such that if $V_0, V_1,\dots$ denote the 0-simplexes of T and $V'_0, V'_1,\dots$ denote ε approximations of these, respectively, then the collection T' to which t belongs if and only if there is a simplex $\sigma=\langle V_{n_i},\dots, V_{n_j}\rangle$ of T such that $t=\langle V'_{n_i},\dots, V'_{n_j}\rangle$ (denote t by σ') is also a finite m-complex such that* (1) *if σ is a k-complex of T then σ' is a k-complex of T',* (2) *if $\sigma_1, \sigma_2 \in T$ and do not intersect, then σ'_1 and σ'_2 do not intersect, and* (3) *if $\sigma_1, \sigma_2, \sigma \in T$ and $\sigma_1 \cap \sigma_2=\sigma$ then $\sigma'_1 \cap \sigma'_2=\sigma'$.*

Proof. The remainder of the theorem is established easily from part (2). Let $\varepsilon>0$ be less than half the distance between any two disjoint simplexes of T.

Suppose that $\sigma_1=\langle V_0,\ldots,V_i\rangle$ and $\sigma_2=\langle W_0,\ldots,W_j\rangle$ are disjoint simplexes of T, but that $P\in\sigma_1'\cap\sigma_2'$. Then $P=\sum_0^i a_pV_p'=\sum_0^j b_pW_p'$, where $\sum_0^i a_p=1$, $a_p\geqq 0$; $\sum_0^j b_p=1$, $b_p\geqq 0$. Thus, we have

$$2\varepsilon < \left\|\sum_0^i a_pV_p-\sum_0^j b_pW_p\right\| = \left\|\sum_0^i a_pV_p-\sum_0^i a_pV_p'+\sum_0^j b_pW_p'-\sum_0^j b_pW_p\right\|$$

$$\leqq \sum_0^i a_p\|V_p-V_p'\|+\sum_0^j b_p\|W_p'-W_p\| \leqq 2\varepsilon,$$

which yields a contradiction.

5. ***m* by *n* transformations.** In this section we prove several theorems about m by n transformations, the first of which says that an $m\times n$ transformation can be considered as a sequence of small steps, with a bound on crossing.

THEOREM 3. *Suppose* f: $[0, 1)\to K$ *and* g: $[0, 1)\to L$ *are knot functions such that* g *has been obtained from* f *by an* m *by* n *transformation* T *and that* D *is a polyhedral disk,* $S\subset[0, 1)$, *arc* $AXB=f(S)$, *arc* $AYB=g(S)$, *and* $\operatorname{Bd}(D)=AXBYA$, *where* D, S, AXB *and* AYB *are as in the definition of m by n transformation. Then, there is a sequence* $g_0, g_1,\ldots, g_q$ *of knot functions such that* (1) $g_0=f$, $g_q=g$, (2) $g_p(x)=f(x)$ *for* $x\in[0, 1)-S$ *and* $g_p(S)\subset D$ *for* $p=0, 1,\ldots, q$, (3) g_p *is obtained from* g_{p-1} *by a simple transformation for* $p=1,\ldots,q$, (4) *if* D_p *is the subdisk of* D *bounded by* $g_0(S)\cup g_p(S)$ *for* $p=1,\ldots,q$, *then* $D_p\subset D_{p+1}$ *for* $p=1,\ldots,q-1$, *and* (5) *if* $c(g_p)$ *denotes the number of crossing points of* $\pi g_p([0, 1))$ *and* c *denotes the number of crossing points of* πL *in* $\operatorname{Int}(\pi D)$, *then* $c(g_p)\leqq 2c+\sup\{c(f), c(g)\}$ *for* $p=0, 1,\ldots, q$.

Proof. We proceed by induction on the number c. Let T_D denote a triangulation of D as guaranteed in the definition of m by n transformation.

Case 1. $c=0$. We note that (1) if there are no components of $\pi(L)\cap\pi(\operatorname{Int}(D))$, or (2) if every component of $\pi(L)\cap\pi(\operatorname{Int}(D))$ has exactly one endpoint on $\pi\operatorname{seg} AXB$ and exactly one endpoint on $\pi\operatorname{seg} AYB$, or (3) there is exactly one component of $\pi(L)\cap\pi(\operatorname{Int}(D))$ whose endpoints are πA and πB, respectively, then T is a transformation of Type 0 and the theorem is trivially true. We therefore proceed in this case by induction on the number of components of $\pi(L)\cap\pi(\operatorname{Int}(D))$, assuming there is at least one such component which does not satisfy the above.

We first consider a slight adjustment of seg AXB into $\operatorname{Int}(D)$ so as to form a new arc AX_1B, where (1) there is a knot function f_1: $[0, 1)\to(K-AXB)\cup AX_1B$ so that $f(x)=f_1(x)$ for $x\in[0, 1)-S$, and (2) if D_1 denotes the subdisk of D bounded by $AYB\cup AX_1B$ and C is a component of $\pi\operatorname{Int}(D_1)\cap\pi f_1([0, 1))$, then C has one endpoint on $\pi\operatorname{seg} AX_1B$ and one endpoint on $\pi\operatorname{seg} AXB$. Thus, there is a Type 0 transformation T_1: $f\to f_1$, and f_1, g, and $\operatorname{Cl}(D-D_1)$ satisfy the same induction hypothesis.

Suppose some component C of $\pi L\cap\pi\operatorname{Int}(D_1)$ has the property that the endpoints P, Q of C lie in πAXB (the case for AYB is handled analogously). It follows that there is such a component $C'=\operatorname{seg} P'Q'$ whose closure separates no other such component from $\pi\operatorname{seg} AYB$ in πD_1. Let TU denote a polygonal arc in

D_1 such that (1) $T, U \in AX_1B$ and seg $TU \subset \text{Int } D_1$, (2) πTU separates C' from all other components of $\pi L \cap \pi \text{ Int } (D_1)$ in πD_1, and (3) there is a knot function $f_2: [0, 1) \to (f_1([0, 1)) - (\text{subarc } TU \text{ of } AX_1B)) \cup TU$ so that $f_2(x) = f_1(x)$ for $x \in [f_1^{-1}(f_1([0, 1)) \cap f_2([0, 1)))]$. We see that f_2 has been obtained from f_1 by transformation T_2 of Type II′ (respectively, Type I, if one of P' or Q' is πA or πB).

We now apply our induction hypotheses to f_2, g and the subdisk of D bounded by $f_2(S) \cup g(S)$ to obtain a sequence $h_0 = f_2, h_1, \ldots, h_t = g$ satisfying the conclusions of the theorem. Our desired sequence is then $f, f_1, h_0, h_1, \ldots, h_t$. Also note that $c(f_1) = c(f_0)$, $c(h_0) = c(f) - i$ where $i = 1$ or 2. Then $c(h_p) \leqq 2c + \sup \{c(h_0), c(h_t)\} \leqq 2c + \sup \{c(f), c(g)\}$, $p = 0, \ldots, t$, and so if our sequence is then labeled $g_0, \ldots, g_{t+2}$, we have $c(g_p) \leqq 2c + \sup \{c(f), c(g)\}$, $p = 0, \ldots, t+2$.

Case 2. $c > 0$. First we construct f_1 and D_1 as in the previous case. There is a component C of $\pi(\text{Int } (D_1)) \cap \pi(L)$ such that $\bar{C}$ contains two arcs EFG and HFJ such that (1) EFG crosses HFJ at F, (2) seg $EFG \cup HFJ$ is open relative to C and (3) $(EFG \cup HFJ) \cap \text{Bd } (\pi D_1) = E$ belongs to seg πAX_1B or to seg πAYB. We handle only the case $E \in \text{seg } \pi AX_1B$, since the other case is handled analogously.

There exist points P, Q, H_1, G_1, J_1 of D such that (1) P and Q occur in the order $APQB$ on arc AX_1B, (2) $\pi H_1 \in \text{seg } HF$ of arc HFJ, $\pi G_1 \in \text{seg } FG$ of arc EFG, and $\pi J_1 \in \text{seg } FJ$ of arc HFJ, (3) there is a polygonal arc PG_1Q containing H_1, G_1, and J_1 and which is a subset of D_1 such that (a) seg $PG_1Q \subset \text{Int } (D_1)$, (b) $\pi(PG_1Q) \cap (\pi L \cap \pi \text{ Int } (D_1)) = \pi(H_1 \cup G_1 \cup J_1)$, and (c) if PQ denotes the subarc of AX_1B with endpoints P, Q, then there is a knot function $f_2: [0, 1) \to (f_1([0, 1)) - PQ) \cup PG_1Q$ so that $f_2(x) = f_1(x)$ for $x \in f_1^{-1}[f_1([0, 1)) \cap f_2([0, 1))]$. We then consider a new knot function f' and a sequence satisfying the conditions of the theorem for f, f_2, the sequence being f, f_1, f', f_2 where the construction of f' may be determined from Figure 1. The dotted interval denotes the change from f_1 to f'. Here there exist transformations $T': f_1 \to f'$ and $T_2: f' \to f_2$, where T' is Type II and T_2 is of Type III.

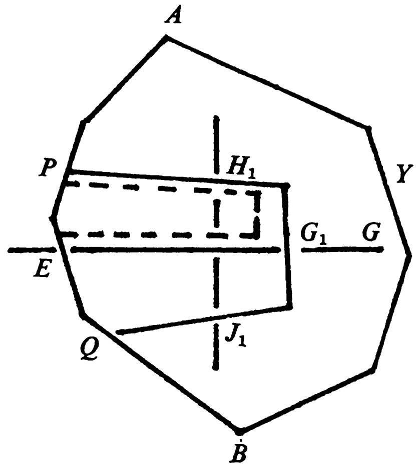

FIGURE 1

Now, using the induction hypothesis on f_2, g and the subdisk of D bounded by $f_2(S) \cup g(S)$, we find a sequence $h_0=f_2, h_1, \ldots, h_t=g$ satisfying the conclusion of the theorem, where $c(h_p) \leqq 2(c-1)+\sup\{c(f), c(g)\} \leqq 2c+\sup\{c(f), c(g)\}$. Also $c(f_2)=2+c(f) \leqq 2c+\sup\{c(f), c(g)\}$, since $c \geqq 1$. Therefore, the desired sequence is $f, f_1, f', f_2, h_1, \ldots, h_t$.

THEOREM 4. *If f: $[0, 1) \to K$ and g: $[0, 1) \to L$ are knot functions and $T: f \to g$ is an $m \times n$ transformation, then there is an orientation-preserving homeomorphism h from E^3 onto E^3 taking K onto L and which is the identity on $K \cap L$.*

Proof. Let arcs AXB, AYB and disk D be in the definition of m by n transformation. Let D' denote a polyhedral disk so that $D-(A \cup B) \subset \text{Int } D'$, $D' \cap K = AXB$ and $D' \cap L = AYB$. Let U denote a polyhedral open set containing $D'-(A \cup B)$ so that $K-D'=L-D' \subset E^3-U$. By Corollary 1 of [12] there is a simplicial isotopy h_t of E^3 onto E^3 which is (1) fixed outside U, (2) is invariant on D', (3) is the identity for $t=0$, and (4) takes AXB onto AYB for $t=1$. The desired mapping is $h_1(x)$.

THEOREM 5. *If f: $[0, 1) \to K$ and g: $[0, 1) \to L$ are knot functions and $T: f \to g$ is an m by n transformation of Type 0, then $C(f)=C(g)$.*

Proof. The proof is omitted.

6. **Simplicial isotopies.** We need to make several new definitions at this stage which are mainly restrictions of the definitions in [12], [13] to E^3.

$H(E^3)$ will denote the space of homeomorphisms of E^3 onto E^3 together with the compact-open topology. An element h of $H(E^3)$ will be said to be *piecewise linear* if there is a rectilinear triangulation T of E^3 such that h is linear on each simplex of T. A rectilinear triangulation is one whose simplexes are points, straight line intervals, triangular disks or solid tedrahedra. Furthermore, if P is a polyhedron in such a triangulation T and f: $|P| \to E^3$ then f will be said to be piecewise linear if f is linear on each simplex of some subdivision of T. If G: $|P| \times I \to E^3$ then G will be called a piecewise linear isotopy if G is continuous and each G_t is a piecewise linear homeomorphism of $|P|$ into E^3. In addition, in the above situation G will be called a simplicial isotopy if G is a piecewise linear isotopy such that G_t is linear on each simplex of some fixed subdivision of P for all t in $[0, 1]$. Also if G: $|P| \times I \to E^3$ is a piecewise linear isotopy and $\varepsilon > 0$, G is called an ε-isotopy if the image of $p \times [0, 1]$ has diameter less than ε for each $p \in |P|$.

We now state a useful theorem due to Moise [6]. The advantage of this theorem is that it allows us to concentrate on the piecewise linear homeomorphisms.

THEOREM M. *If some homeomorphism ϕ of 3-space E^3 upon itself preserves the orientation and transforms a given simple closed polygon J into a given simple closed polygon J', then there is a piecewise linear homeomorphism with these properties.*

THEOREM 6. *Suppose that each of K and L is a polygonal knot in regular position in E^3 and that f is an orientation-preserving piecewise linear homeomorphism from E^3 onto E^3 such that $f(K)=L$. Then, there is a simplicial isotopy $\phi: K\times[0, 1]\rightarrow E^3$ such that $\phi(x, 0)=x$ and $\phi(x, 1)=f(x)$ for $x\in K$.*

Proof. This theorem has been proved by Graeub [6]. It may also be shown using the work of Sanderson in [12], [13], [14].

THEOREM 7. *Suppose K, L, f and ϕ are as in Theorem 6 and that S is a geometric 2-sphere so that $\phi(K\times[0, 1])\subset \operatorname{Int} S$. Then,* (1) *if R is a countable subset of* [0, 1] *and U is an open subset of S, there is a plane P tangent to S at some point A of U such that if $t\in R$ then $\phi_t(K)$ is in regular position relative to P, and* (2) *there is a fixed integer N such that if Q is a plane tangent to S and $\phi_t(K)$ is in regular position relative to Q, then the projection $\pi_Q(\phi_t(K))$ has no more than N crossings.*

Proof. (1) For each $\phi_t(K)$ there is a closed nowhere dense subset C_t of S such that $\phi_t(K)$ is in regular position relative to the tangent plane P of S if and only if P is tangent to S at a point of $S-C_t$. Therefore, since R is countable and S is a Baire space, there is a point A of U such that if P is a plane tangent to S at A and $t\in R$, then $\phi_t(K)$ is in regular position relative to P.

(2) Since ϕ is a simplicial isotopy there is a triangulation T of K with exactly n 1-simplexes such that if $t\in[0, 1]$ and s is a 1-simplex in T then $\phi_t(s)$ is a 1-simplex in $\phi_t(K)$. If s and s' are two 1-simplexes in T then $\pi_Q(\phi_t(s))$ and $\pi_Q(\phi_t(s'))$ can cross each other no more than once, so $\pi_Q(\phi_t(K))$ has no more than $N=n^2$ crossing points.

7. **Simple sequences.** We now combine the results of some of the previous sections to prove the following:

THEOREM 8. *Suppose that each of $g: [0, 1)\rightarrow K$ and $h: [0, 1)\rightarrow L$ is a knot function, and that f is an orientation-preserving homeomorphism from E^3 onto E^3 such that $f(K)=L$, and $f(g)$ and h induce the same orientation on L. Then, there is a simple sequence of knot functions*

$$g = g_0 \xrightarrow{S_1} g_1 \xrightarrow{S_2} \cdots \xrightarrow{S_n} g_n = f(g),$$

and thus a simple sequence of boundary collections

$$C(g) = C(g_0) \xrightarrow{S_1} C(g_1) \xrightarrow{S_2} \cdots \xrightarrow{S_n} C(g_n) = C(h).$$

Proof. We apply Theorem 6 to obtain a simplicial isotopy $\phi: K\times[0, 1]\rightarrow E^3$ such that $\phi(x, 0)=x$ and $\phi(x, 1)=f(x)$. Let T denote a triangulation of K so that if $S\in T$ and $t\in[0, 1]$ then ϕ_t is affine on S.

In order to simplify the statements involved in our proof we suppose without loss of generality that $\phi(K\times[0, 1])\subset\operatorname{Int} S$, where S is a geometric 2-sphere tangent to the xy plane at (0, 0, 0) and where if $(x, y, z)\in\operatorname{Int}(S)$ then $z>0$. There is an open subset U of S containing (0, 0, 0) so that if $A\in U$ and P is a plane tangent to

S at A, then K and L are in regular position relative to P. Let R denote the rationals in $[0, 1]$.

By Theorem 2 and the compactness of $[0, 1]$, there is an $\varepsilon > 0$ so that if $t \in [0, 1]$ and $T_t = \{\phi(s, t) : s \in T\}$ then the conclusions of Theorem 2 follow for 2ε approximations of the 0-simplexes of T_t. Let $0 < d < 1$ be such that if $t, t' \in [0, 1]$ and $|t - t'| < d$ and S is a 0-simplex of T, then $\|\phi_t(S) - \phi_{t'}(S)\| < \varepsilon$.

We apply Theorem 7 to g, h, ϕ, R, S, U to find $A' \in U$ so that (1) E^3 may be rotated slightly about the center of S so as to move A' to $(0, 0, 0)$ and so that no 0-simplex of T_i $(i = 0, 1)$ is moved as much as $\varepsilon/2$, and (2) if P denotes the plane tangent to S at A' and $t \in R$, then $\phi_t(K)$ is in regular position relative to P. At this stage we assume for the moment that the point A' involved is $(0, 0, 0)$ itself.

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be a partition of $[0, 1]$ so that (1) each $a_p \in R$ and (2) if $0 \leqq p < k$, then $|a_p - a_{p+1}| < d$. Let $V_0, \ldots, V_q$ denote the 0-simplexes of T in some cyclic order, and for each $p \in \{0, \ldots, k\}$ let $A_{0p}, \ldots, A_{qp}$ denote $\varepsilon/2$ approximations of $\phi(V_0, a_p), \ldots, \phi(V_q, a_p)$, respectively, such that if J_p is the union of intervals $A_{0p}A_{1p}, A_{1p}A_{2p}, \ldots, A_{qp}A_{0p}$ then (0) $A_{ip} = \phi(V_i, a_p)$ for $p = 0, k$, (1) each J_p is in regular position, and (2) if $0 \leqq p < k$ then J_p and J_{p+1} do not intersect and no point of $\{A_{0p}, \ldots, A_{qp}, A_{0,p+1} \cdots A_{q,p+1}\}$ projects onto a double point of

$$\pi(J_p \cup J_{p+1}).$$

Now suppose $0 \leqq p < k$ and let $A_i = A_{ip}$, $B_i = A_{i,p+1}$ for $i = 1, \ldots, q$. Letting $J_A = J_p$ and $J_B = J_{p+1}$, the natural p.l. homeomorphisms $h_A \colon \phi(K, a_p) \to J_A$, $h_B \colon \phi(K, a_{p+1}) \to J_B$ induce knot functions $g_A = h_A \phi_{a_p} g$ and $g_B = h_B \phi_{a_p+1} g$.

We now consider a knot function $g' \colon [0, 1) \to (J_A - A_1A_2) \cup A_1B_1 \cup A_2B_1$ such that $g'(x) = g_A(x)$ for $x \in [0, 1) - g_A^{-1}(A_1A_2)$. We now find that the hypothesis of Theorem 3 holds for g_A, g' and the disk $A_1A_2B_1$, and obtain a sequence of functions $g_A = h_0, h_1, \ldots, h_m = g'$ as in the conclusion of Theorem 3. We then consider the function g', a knot function g'' so that image

$$g'' = (J_A - A_1A_2) \cup (A_1B_1 \cup B_1B_2 \cup A_2B_2),$$

and the disk $B_1B_2A_2$. Again, we obtain a sequence as in Theorem 3. The next disk considered is $A_2B_2A_3$, etc. Thus we obtain a sequence of knot functions $k_0 = g_A$, $k_1, \ldots, k_r = g_B$ so that k_{i+1} is obtained from k_i by a simple transformation. If we do this for each a_p, $0 \leqq p < k$, then it is evident that we can produce a sequence $g_0', \ldots, g_n'$ as in the conclusion of the theorem. Remembering now that the point A' involved previously was not necessarily the origin, we see from the following how the sequence $g_0', \ldots, g_n'$ defined above can be used to define a new sequence $g_0, \ldots, g_n$, where the boundary collections are the usual ones relative to the xy plane.

We use a small rotation V of E^3 so that the center of S is fixed, A' is moved to the origin, and no 0-simplex of T_i $(i = 0, 1)$ is moved as much as $\varepsilon/2$. We let $g_p = Vg_p'$, $p = 0, \ldots, n$, and let $v(K) = K'$ and $v(L) = L'$. Now using new approximations of

0-simplexes new knots K'' and L'' may be determined so that (1) K'' is near K, K' and L'' is near L, L' and (2) some new sequences of knot functions may be defined as in the case of J_A and J_B above for the pairs (K, K''), (K'', K'), (L', L'') and (L'', L). The knot functions obtained in these four cases are used together with $g_0, \ldots, g_n$ to give the desired sequence.

8. **Matrix representations.** In this section we give a way of associating with a knot function f: $[0, 1) \rightarrow K$ an $n+2$ by $2n$ matrix $M(K)$, where πK has n double points and each entry of $M(K)$ is a 0, 1 or -1.

We start labeling the columns across the top using the word $W(f)$ of f. The rows represent in a one-to-one way the elements of the boundary collection, $C(f)$ or equivalently, the complementary domains. For example, if row i corresponds to $C_1 \in C(f)$ and $B=(\pi f(a_p))^{e_p}(\pi f(a_{p+1}))^{e_{p+1}} \in C_1$ then $M_{iq}=e_q$ for $q=p, p+1$. Any other places in row i not filled in this way are filled with zeros. A matrix for the trefoil knot T of Figure 1 is

$$\begin{matrix} & a & b^{-1} & c & a^{-1} & b & c^{-1} \end{matrix}$$
$$M = \begin{Bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{Bmatrix}.$$

Also, it seems to be convenient to think of two of the matrices as being equivalent if one is obtained from the other by

(0) the identity transformation,
(1) any cyclic permutation of columns,
(2) reversal of all signs,
(3) any permutation of rows,
(4) reversal of order of columns, or
(5) any operation corresponding to a simple transformation.

With the aid of Theorems 1 and 4, it is not hard to show

THEOREM 14. *If $M(K)$ is equivalent to $M(L)$ then K is isomorphic to L.*

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